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A GENERALISATION OF A SOLUTION CONCEPT FOR THE LINEAR PROGRAMMING PROBLEM WITH INTERVAL COEFFICIENTS

A generalisation of the known concept of solving linear programming problems with interval coefficients is proposed. The generalisation allows the decision maker to make a better final decision, as he will have much more information about the problem under consideration. The algorithm proposed for determining the solution makes use of linear programming methods only.

1. Introduction

In many practical applications of linear programming the problem coefficients cannot be determined in a precise way. That is why quite a few researchers have been trying to propose a way to solve linear programming problems with imprecise coefficients [2]–[5], [7], [8], [10]. The difficulty lies in the fact that while dealing with such problems it cannot be unequivocally what the optimal solution is. Its understanding depends strongly on the decision maker preferences and attitude. For this reason it is necessary to consider various concepts, so that each decision maker can choose one that suits him most.

In this paper, we propose a generalisation of one of the concepts of solving linear programming problems with interval coefficients – the one proposed in [10] and developed in [8]. The generalisation allows the decision maker to have more information about his problem with imprecise coefficients and the possible form of its optimal solution.

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2. Formulation of the problem

We consider the following problem:

$$\sum_{j=1}^{n}\left[\underline{c}_j,\overline{c}_j\right]\cdot x_j \to \min ,$$

$$\sum_{j=1}^{n}\left[\underline{a}^i_j,\overline{a}^i_j\right]\cdot x_j \geq \left[\underline{b}^i,\overline{b}^i\right], \quad i = 1,\ldots, m , \tag{1}$$

$$x_j \geq 0\left(j=1, ..., n\right).$$

The upper and lower ends of the intervals can be equal, so that model (1) comprises also the case where some coefficients are precise. Arithmetical operations on interval numbers are understood in a standard way ([9]), i.e., they can be performed at the lower and upper ends respectively, so that the left hand and the right hand sides of expressions in (1) are also closed intervals.

3. Solution concept proposed in the literature and its drawbacks

In [8] and [10], the authors propose to consider the following notions:

• the largest possible feasible region, determined in the case of positive decision variables by the following system of constraints:

$$\sum_{j=1}^{n}\overline{a}^i_j\cdot x_j \geq \underline{b}^i, \quad i = 1, \ldots, m, \; x_j \geq 0\left(j=1, ..., n\right) \tag{2}$$

• the smallest possible feasible region, determined in the case of positive decision variables by the following system of constraints:

$$\sum_{j=1}^{n}\underline{a}^i_j\cdot x_j \geq \overline{b}^i, \quad i = 1, \ldots, m, \quad x_j \geq 0\left(j=1, ..., n\right) \tag{3}$$

• the most favourable objective function, corresponding in the case of positive decision variables to the following objective function:

$$\sum_{j=1}^{n}\underline{c}_j\cdot x_j \to \min \tag{4}$$

• the least favourable objective function, corresponding in the case of positive decision variables to the following objective function:

$$\sum_{j=1}^{n} \bar{c}_j \cdot x_j \to \min . \tag{5}$$

Then they suggest to solve problem (1) by means of solving the following two crisp linear programming problems:

• the Best Optimum Problem: objective function (4), feasible region (2),

• the Worst Optimum Problem: objective function (5), feasible region (3).

Such a solution informs the decision maker about two extreme alternatives:

• what the optimum will be in case the unknown coefficients (all at the same time) take on the most favourable values,

• what the optimum will be in case the unknown coefficients (all at the same time) take on the least favourable values.

This approach yields interesting information, but the quality of this information might be improved. There are at least two reasons justifying the need to give the decision maker more exact information about the solution.

First reason lies in the fact that the Worst Optimum Problem may be infeasible, in which case the decision maker does not really know what range of objective function values he can expect. Let us illustrate this statement by means of an example:

Example 1

$$\begin{aligned}
&-x_1 + 5x_2 \to \min \\
&[3{,}5] \cdot x_1 + [2{,}4] \cdot x_2 \geq [3{,}8] \\
&-x_1 \geq -1; -x_2 \geq -1; \\
&x_1, x_2 \geq 0
\end{aligned}$$

The following table shows the results:

Best Optimum Problem: $-x_1 + 5x_2 \to \min$ $5 \cdot x_1 + 4 \cdot x_2 \geq 3$ $-x_1 \geq -1; -x_2 \geq -1;$ $x_1, x_2 \geq 0$	Worst Optimum Problem: $-x_1 + 5x_2 \to \min$ $3 \cdot x_1 + 2 \cdot x_2 \geq 8$ $-x_1 \geq -1; -x_2 \geq -1;$ $x_1, x_2 \geq 0$
Solution: $x_1 = 1, x_2 = 0;$ Objective function = –1	Solution: the problem is infeasible

The above information does not tell the decision maker much about his problem. In the modification of the approach proposed later on the information about the solution given to the decision maker will be much more detailed.

The other reason for which the approach from [8] and [10] should be modified is a possible different interpretation of the imprecision in the problem coefficients. The imprecision, as is often the case, may be due to the fact that it is not the decision maker that fixes the coefficients and he does not have enough information about their values – they will be known only in the future, and the decision has to be taken now. But there is also another possible source of imprecision: the coefficients are expressed as intervals also in case it is the decision maker that fixes the coefficients. He can fix them within the manoeuvre possibilities limited by the intervals and wants to see what values he should choose so that he is most satisfied with the optimal value of the objective function, on the one hand, and with the choice of the coefficient values on the other hand. Indeed, the choice of coefficient values, is not indifferent to him, e.g., it is on the way he fixes the technical and the right hand side coefficients that the quality of the products may depend – the higher the requirements, the better the quality. The question of the various ways of interpreting interval coefficients is discussed, e.g., in [6].

The modification proposed in the following will constitute an improvement in both respects compared to the original method.

4. Modification of the approach

We propose to consider for problem (1) the following family of linear programming problems P_λ $(\lambda \in [0,1])$: :

$$\sum_{j=1}^{n} C_\lambda\left(\underline{c}_j, \overline{c}_j\right) \cdot x_j \to \min$$

$$\sum_{j=1}^{n}\left[\overline{a}_j^i - \lambda\left(\overline{a}_j^i - \underline{a}_j^i\right)\right] \cdot x_j \geq \underline{b}^i + \lambda\left(\overline{b}^i - \underline{b}^i\right), \quad i = 1, \ldots, m \tag{6}$$

$$x_j \geq 0 \left(j = 1, \ldots, n\right)$$

where C_λ is a function $\Re^2 \to \Re$ with parameter λ. The function C_λ can be chosen by the decision maker. But before discussing the objective function let us explain the constraints of (6).

If we denote by FR_λ the feasible region of (6) for λ, then the following lemma is true and its proof is straightforward, because the decision variables are positive.

Lemma 1. If $\lambda_1 \geq \lambda_2$ then $FR_{\lambda_2} \subseteq FR_{\lambda_1}$ $(\lambda_1, \lambda_2 \in [0,1])$

What is more, FR_0 is identical with the largest possible region mentioned above and FR_1 – with the smallest possible region. Thus, if family P_λ $(\lambda \in [0,1])$ is considered, intermediate forms of the feasible region – and not only the two extreme ones – are taken into account.

The parameter λ can be interpreted as the degree of requirements set to the solution. Indeed, λ = 0 corresponds to the case when the constrains are the least requiring (there are thus many feasible solutions) and λ = 1 to the other extreme case – when the constrains are most requiring (here we have the smallest possible feasible region).

Let us now discuss the function C_λ. The decision maker can choose it according to his preferences. Here are four proposals for it:

a) $C_\lambda(s,t) = \min\{s,t\}$

b) $C_\lambda(s,t) = \max\{s,t\}$

c) $C_\lambda(s,t) = \max\{s,t\} - \lambda(\max\{s,t\} - \min\{s,t\})$

d) $C_\lambda(s,t) = \min\{s,t\} + \lambda(\max\{s,t\} - \min\{s,t\})$

It would be reasonable to consider cases a) and b) jointly. They might be used when the decision maker would like to see the best possible optimum and the worst possible optimum for each of the problems P_λ $(\lambda \in [0,1])$. Cases a) and b) should be used above all when the decision maker has no influence on the values of coefficients.

Case c) might be used when the decision maker can choose λ and if the following dependence is true: the higher the requirements set for the solution, the lower the objective function coefficients. Such a dependence may be true in some real world situations. Higher requirements imposed on the solution are the result of choosing lower technical coefficients and a greater right hand side of constraints. Lower technical coefficients representing, e.g., the use of some resource per product unit, may lead to lower cost per product unit, and the unit cost may be represented by the objective function coefficients.

Case d) will also be used when the decision maker can choose λ, in the case where the following dependence is true: the higher the requirements set for the solution, the greater the objective function coefficients. Again, this may be true in some applications, e.g., if the technical coefficients represent the use of a resource that has some influence on the quality of the product. In this case lower technical coefficients mean greater cost resulting from lower quality.

Other types of function C_λ can be considered, too.

The basic idea of our approach is that the decision maker chooses function C_λ and then solves problems P_λ $(\lambda \in [0,1])$ for some values of λ. This would give the decision maker an overview of possible optimal objective function values. This will be useful in both cases of imprecision source:

a) when the coefficients are unknown and the decision maker has no influence on them: here the proposed approach would give the decision maker an overview of what range of objective function values is possible;

b) when the decision maker can choose the coefficients himself, e.g., by selecting the parameter λ and choosing one P_λ, the approach would help him to make his choice. He would see what objective function values correspond to which degree of requirements and would be able to decide whether he has to choose very high requirements or can accept lower ones, because setting lower (but acceptable) requirements gives him a better optimal objective function value.

In case a), if the decision maker is interested just in the range of possible objective function values, it may not be necessary to solve many problems P_λ. In fact, it may be argued that just the extreme cases are necessary – which would almost bring us back to the original proposal from [8] and [10]. However, as Example 1 shows, the original approach may not indicate where the second extreme case (the upper bound of the range of possible objective function values) exactly lies. Here our approach would give an answer, using an algorithm which allows us to avoid solving $P_\lambda(\lambda \in [0,1])$ for many values of λ.

5. Algorithm for determining the range of possible optimal objective function values

The following algorithm will determine the range of possible optimal objective function values when C_λ in problems $P_\lambda(\lambda \in [0,1])$ corresponds to case a) or b) (or generally does not depend on λ). The algorithm is based on the halving procedure used, e.g., in [4]. ε is an accuracy parameter and should be chosen beforehand.

Algorithm:

Step 1: Set $\lambda := 0$.

Step 2: Solve P_λ. If P_λ is infeasible – **STOP**, no feasible solution is possible. Otherwise set MIN equal to the current value of the objective function and set $\alpha := \lambda$.

Step 3: Set $\lambda := 1$.

Step 4: Solve P_λ. If P_λ is feasible, set MAX equal to the current value of the objective function and **STOP** – the range of the objective function values is [MIN, MAX]; 1 is the highest value of parameter λ for which the problem P_λ is feasible.

Step 5: Set $\beta := \lambda$.

Step 6: Set $\lambda = {(\alpha+\beta)}/{2}$.

Step 7: Solve P_λ . If P_λ is feasible, set MAX equal to the current value of the objective function, set $\alpha := \lambda$. Otherwise set $\beta := \lambda$.

Step 8: If $\beta - \alpha \geq \varepsilon$ go to Step 6. Otherwise **STOP** – the range of the objective function values is [MIN, MAX], α is the highest value of parameter λ for which the problem P_λ is feasible.

The effect of the algorithm will now be presented using Example 1. In this example any of the proposed choices of function C_λ would lead to function $C_\lambda(t,t) = t$. Let us assume ε = 0.01.

Step 1	$\lambda = 0$
Step 2	MIN = –1, $\alpha = 0$
Step 3	$\lambda = 1$
Step 4	P_1 is infeasible
Step 5	$\beta = 1$
Step 6	$\lambda = 0.5$
Step 7	MAX = 1.5; $\alpha = 0.5$
Step 8	$\beta - \alpha = 0.5 \geq \varepsilon$
Step 6	$\lambda = 0.75$
Step 7	$P_{0.75}$ is infeasible. $\beta = 0.75$
Step 8	$\beta - \alpha = 0.25 \geq \varepsilon$
Step 6	$\lambda = 0.625$
Step 7	MAX = 3.318; $\alpha = 0.625$
Step 8	$\beta - \alpha = 0.125 \geq \varepsilon$
Step 6	$\lambda = 0.6875$
Step 7	$P_{0.6875}$ is infeasible. $\beta = 0.6875$
Step 8	$\beta - \alpha = 0.0625 < \varepsilon$ – STOP

Thus, the decision maker knows that whenever the problem has a solution, the optimal value of the objective function will lie in the range [–1, 3.318]. The largest value of the parameter λ for which P_λ has a solution is equal to 0.625, and for this λ the optimum is equal to 3.318. For the other extreme, $\lambda = 0$, the optimum is equal to –1. This is clearly a more exact information about the solution of the problem than the one given by the approach from [8] and [10], as shown in Example 1.

If the decision maker can choose the value of λ (the degree of requirements) himself, he can look for a compromise solution. He knows that if the requirements are very high, there may be no solution, and the higher the requirements, the less interesting the objective function is. So, high requirements may would lead to good solutions from the point of view of constraints, but with a bad objective function. A compromise would be thus needed. In our example, the decision maker may decide that it is enough for him that the objective function does not exceed 0. From the above solution he knows that this is obtained for some λ from the interval [0, 0.625]. Even by a trial and error method, he will then be able to find such λ, i.e., such a degree of requirements, that corresponds to the optimal objective function value equal to 0. This value is approximately equal to 0.3. If this level of requirements satisfies the decision maker, he will accept $\lambda = 0.3$.

Of course, here we might also use a more formal approach, using the idea from [1]. We might define a function $G(\lambda)$ expressing the satisfaction of the decision maker with the optimal value of the objective function from problem P_λ and define a compromise solution as one corresponding to λ for which function $\min\{G(\lambda), 1-\lambda\}$ is maximal.

6. Conclusions

In many practical situations where linear programming models are used the coefficients of the model cannot be determined in a precise way. Sometimes they can only be given in the form of interval numbers. In this paper, we present a method of solving linear programming problems with interval coefficients that has been presented in the literature. Then we indicate some of its drawbacks and propose a modification which, in our opinion, is a simple, but useful improvement of the method. In some cases the information that the decision maker obtains in the modified method would be significantly more thorough than in the original one. The modification also allows for much more freedom in the choice of some parameters – it is in fact a family of methods, comprising the original method as its special case. And this gain does not require much more computational effort – the solution algorithm makes use of linear programming methods only.

Further research is needed to consider cases of non-positive or unrestricted decision variables. And generally, the needs of the decision makers as to the form of the solution of a linear programming problem with interval coefficients should be further explored, so that new concepts can be built up that will correspond to the preferences and attitudes of various decision makers.

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Uogólnienie pewnej koncepcji rozwiązania zadania programowania liniowego z przedziałowymi współczynnikami

W pracy rozważa się zadanie programowania liniowego z przedziałowymi współczynnikami po obu stronach ograniczeń i w funkcji celu. Zanalizowano znaną metodę rozwiązywania tego problemu, w której decydent otrzymuje informację o dwóch ekstremalnych przypadkach: o optimum dla przypadku, kiedy wszystkie współczynniki przyjmują najbardziej niekorzystne wartości i dla przypadku, kiedy przyjmują one najmniej korzystne wartości. Ta informacja nie jest bardzo przydatna, jeśli jeden z tych przypadków prowadzi do problemu sprzecznego – wtedy decydent nie ma żadnej informacji o zakresie możliwych wartości funkcji celu. Proponuje się metodę (i odpowiedni algorytm, wykorzystujący tylko metody programowania liniowego), która w każdym przypadku pozwala uzyskać informację o zakresie możliwych wartości funkcji celu.